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On Cubic Curves.

By F. Franklin.

The following discussion of some points in the theory of cubics is given on account of the method by which the results are deduced; most, if not all, of the theorems obtained are known. I take the system of cubics $x^3 + y^3 + z^3 + 6lxyz = 0$, and obtain easily and consecutively a number of results, some relating to single cubics and some to the system, from the consideration that the mixed polars of a pair of points with respect to all the cubics pass through a fixed point, and that the polar lines and polar conics of a point likewise pass through one and four fixed points respectively. These facts are obvious, since the equations of these polar derivatives contain linearly a single indeterminate, l.

The system of cubics, the system of mixed polars of a given pair of points, 1 and 2, the system of polar conics of a given point, the system of polar lines of a given point, are given by the equations

$$x^3 + y^3 + z^3 + 6lxyz = 0 \tag{1}$$

$$x_1x_2x + y_1y_2y + z_1z_2z + l\{(y_1z_2 + y_2z_1)x + (z_1x_2 + z_2x_1)y + (x_1y_2 + x_2y_1)z\} = 0 \tag{2}$$

$$x_1x^2 + y_1y^2 + z_1z^2 + 2l(x_1yz + y_1zx + z_1xy) = 0 \tag{3}$$

$$x_1^2x + y_1^2y + z_1^2z + 2l(y_1z_1x + z_1x_1y + x_1y_1z) = 0. \tag{4}$$

Through every point in the plane (not a common point of the system in question) there passes one and only one member of each of the above systems. It is to be remembered that the mixed polar of two points is the polar line of either with respect to the polar conic of the other; and it is plain that if the mixed polar of A and B with respect to any cubic passes through C, the mixed polar of A and C passes through B, and in particular that if C is the fixed point on the mixed polars of A and B, B is the fixed point on the mixed polars of A and C.

First let us inquire whether the fixed point on the mixed polars of a pair of points, 1 and 2, lies upon the line joining 1 and 2. The necessary and sufficient condition that it should, is that when the mixed polar is made to pass through 1, it shall also pass through 2.

Putting in equation (2) for x, y, z successively x_1, y_1, z_1 and x_2, y_2, z_2, we obtain

$$(5)\qquad \begin{aligned} x_1^2x_2 + y_1^2y_2 + z_1^2z_2 + 2l(y_1z_1x_2 + z_1x_1y_2 + x_1y_1z_2) = 0 \\ x_2^2x_1 + y_2^2y_1 + z_2^2z_1 + 2l(y_2z_2x_1 + z_2x_2y_1 + x_2y_2z_1) = 0\,, \end{aligned}$$

which are to be satisfied by the same value of l. Hence the required condition is

$$0 = \begin{vmatrix} x_1^2x_2+y_1^2y_2+z_1^2z_2, & y_1z_1x_2+z_1x_1y_2+x_1y_1z_2 \\ x_2^2x_1+y_2^2y_1+z_2^2z_1, & y_2z_2x_1+z_2x_2y_1+x_2y_2z_1 \end{vmatrix} = (x_1^3+y_1^3+z_1^3)x_2y_2z_2 - (x_2^3+y_2^3+z_2^3)\,x_1y_1z_1.$$

But this is obviously also the condition that the two points shall lie on the same cubic of the system. Hence, in order that the fixed point on the mixed polar of 1 and 2 should be on their junction, it is necessary and sufficient that the points 1 and 2 lie on the same cubic of the system. But if 3 is the fixed point on the mixed polars of 1 and 2, 2 is the fixed point on the mixed polars of 1 and 3; hence, by what has just been proved, 3 also is on the same cubic as 1; therefore 3 is the connective of 1 and 2.

It follows from the above, that the connective of two given points, 1 and 2, is the common intersection of the lines

$$x_1x_2x + y_1y_2y + z_1z_2z = 0 \tag{a}$$

$$(y_1z_2 + y_2z_1)x + (z_1x_2 + z_2x_1)\,y + (x_1y_2 + x_2y_1)z = 0 \tag{b}$$

$$(y_1z_2 - y_2z_1)x + (z_1x_2 - z_2x_1)\,y + (x_1y_2 - x_2y_1)z = 0\,, \tag{c}$$

the last of these being the junction of 1 and 2. Or, taking the sum and difference of (b) and (c), the connective is the common intersection of the lines

$$x_1x_2x + y_1y_2y + z_1z_2z = 0 \tag{a'}$$

$$y_1z_2x + z_1x_2y + x_1y_2z = 0 \tag{b'}$$

$$y_2z_1x + z_2x_1y + x_2y_1z = 0\,. \tag{c'}$$

From (b') and (c') we obtain for the coordinates of the connective

$$x:y:z = \left\|\begin{matrix} y_1z_2 & z_1x_2 & x_1y_2 \\ y_2z_1 & z_2x_1 & x_2y_1 \end{matrix}\right\| = x_1^2y_2z_2 - x_2^2y_1z_1 : y_1^2z_2x_2 - y_2^2z_1x_1 : z_1^2x_2y_2 - z_2^2x_1y_1\,,$$

which are the values given by Professor Sylvester, *Amer. Jour. Math.* Vol. III, p. 62; while from (a) and (c) we obtain the values given by Cauchy, viz.:

$$x:y:z = \left\|\begin{matrix} x_1x_2 & y_1y_2 & z_1z_2 \\ (y_1z_2) & (z_1x_2) & (x_1y_2) \end{matrix}\right\|$$

When 1 and 2 coincide, their connective becomes the tangential, and equations (a) and (b) become

$$x_1^2x + y_1^2y + z_1^2z = 0$$

$$y_1z_1x + z_1x_1y + x_1y_1z = 0$$

whence

$$x:y:z = x_1(y_1^3 - z_1^3):y_1(z_1^3 - x_1^3):z_1(x_1^3 - y_1^3).$$

We have found that if two points lie on the same cubic, the fixed point on their mixed polars is their connective; and thus (since when the two points coincide, the mixed polar becomes the polar line) that the fixed point on the polar lines of a given point is the tangential of that point. The four fixed points on the polar conics of a given point are evidently points which have the given point for the fixed point on their polar lines; therefore by what precedes, they are on the cubic on which the given point lies, and have the given point for their tangential. In particular, it may be noted that the degenerate polar conics of a point are the three pairs of lines obtained by joining the four anti-tangentials of the point.

The condition that two points shall have the same tangential (which includes the condition that they shall lie on the same cubic) is the simultaneous satisfiability of the four equations

$$x_1^2x + y_1^2y + z_1^2z = 0, \quad y_1z_1x + z_1x_1y + x_1y_1z = 0;$$
$$x_2^2x + y_2^2y + z_2^2z = 0, \quad y_2z_2x + z_2x_2y + x_2y_2z = 0.$$

Combining the first equations of these pairs, we find

$$x:y:z = (y_1z_2)(y_1z_2 + y_2z_1):(z_1x_2)(z_1x_2 + z_2x_1):(x_1y_2)(x_1y_2 + x_2y_1);$$

and combining the second equations

$$x:y:z = (y_1z_2)x_1x_2:(z_1x_2)y_1y_2:(x_1y_2)z_1z_2;$$

and thus the condition that two points shall have a common tangential, *i. e.* that they shall be "corresponding" points in Maclaurin's sense, is

$$x_1x_2:y_1y_2:z_1z_2 = y_1z_2 + y_2z_1:z_1x_2 + z_2x_1:x_1y_2 + x_2y_1.$$

But this is also the condition that the mixed polar of the two points with respect to every cubic of the system shall be a fixed line; *i. e.* that the polar of either with respect to the system of polar conics of the other shall be a fixed line; so that each is one of the three double points in the system of polar conics of the other. Hence we have the theorem that if two points, 1 and 2, have the same tangential, each is the double point of a polar conic of the other, and conversely. Moreover, if 3 and 4 are the other two anti-tangentials of the tangential of 1, they are the other two double points of the system of polar conics of 1, and the polar of 2 with respect to any of these conics is $\overline{34}$; but we have seen that this polar, being the mixed polar of 1 and 2, passes through their connective: whence the theorem that if 1234 be the anti-tangentials of any point on the cubic, $\overline{12}$ and $\overline{34}$ intersect on the cubic. Of the four points, any two are a pair of corresponding points, and the other two may be called the complementary pair of corresponding points; and we have found that the junction of a pair of corresponding points is the mixed polar of the complementary pair, and that the junctions of two complementary pairs constitute the polar conic, with respect to some cubic, of the common tangential of the four points. Hence any centre of the quadrangle 1234 is the double point of a polar conic of the common tangential, so that the three centres and the common tangential form again a system of corresponding points.—It is to be observed that we have now a special geometrical interpretation of the equations

$$x_1x_2x + y_1y_2y + z_1z_2z = 0, \quad (y_1z_2 + y_2z_1)x + (z_1x_2 + z_2x_1)y + (x_1y_2 + x_2y_1)z = 0$$

in the case when 1 and 2 are corresponding points (and when consequently these two lines coincide); namely, if 3 and 4 are the complementary pair of corresponding points, the above is the equation of $\overline{34}$, the line which, with $\overline{12}$, forms one of the degenerate polar conics of the common tangential.

We have found the necessary and sufficient condition that the mixed polar of a pair of points with respect to some cubic of the system should be the line joining the points to be that the two points lie on the same cubic; but the algebraical expression of this condition, viz. the coexistence of equations (5), is the same as that of the condition that each point should be, with respect to some cubic of the system, an intersection of the polar line and polar conic of the other; hence we see that the intersections of the polar line and polar conic of a point with respect to *any* cubic of the system lie on the cubic passing through the point. Thus, then, denoting by A the cubic passing through 1 and by X a variable cubic of the system, we have found that of the six intersections with A of the polar conic of 1 with respect to X, four are fixed and two variable; of the three intersections with A of the polar line of 1 with respect to X, one is fixed and two variable; and the two variable intersections of the polar conic coincide with the two variable intersections of the polar line.

We have just found that the intersections of the polar line and polar conic of a point with respect to *any* cubic of the system lie on the cubic passing through the point; or in other words that the contacts of tangents drawn from a point to any of its polar conics lie on the cubic passing through the point. But the system of polar conics of a point with respect to the cubics is simply a system of conics through four fixed points; starting from such a system, then, instead of from the system of cubics, we may state some of the foregoing results as follows: given a system of conics through four fixed points, the locus of the points of contact of tangents to these conics from any point P is a cubic which passes through P, through the four fixed points, and through the three centres of the complete quadrangle formed by these four points; the tangents to this cubic at the four fixed points pass through P, and the tangents at the three centres and at P meet in a point on the cubic.

To find the locus directly, take for the triangle of reference that formed by the three centres of the quadrangle; then the equation of the system may be taken to be

$$x^2 - y^2 + \lambda(x^2 - z^2) = 0,$$

and the polars of any point α, β, γ are

$$\alpha x - \beta y + \lambda(\alpha x - \gamma z) = 0.$$

It may be noted in passing that the fixed point on these lines—which we have seen, is the tangential of α, β, γ with respect to the cubic locus sought—is $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

Eliminating λ between the above two equations, we find for the locus of the points of contact of tangents from α, β, γ to the conics,

$$\begin{vmatrix} x^2 - y^2 & x^2 - z^2 \\ \alpha x - \beta y & \alpha x - \gamma z \end{vmatrix} = 0,$$

i. e. the cubic

$$x^2(\beta y - \gamma z) + y^2(\gamma z - \alpha x) + z^2(\alpha x - \beta y) = 0.$$

For this cubic, we have

$$S = -(\alpha^4 + \beta^4 + \gamma^4) + \beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2$$
$$T = -4\{2(\alpha^6 + \beta^6 + \gamma^6) + 12\alpha^2\beta^2\gamma^2 - 3(\beta^4\gamma^2 + \beta^2\gamma^4 + \gamma^4\alpha^2 + \gamma^2\alpha^4 + \alpha^4\beta^2 + \alpha^2\beta^4)\}.$$

The discriminant is $T^2 + 64S^3$. Writing $\alpha^2, \beta^2, \gamma^2 = f, g, h$ and observing that

$$f^2 + g^2 + h^2 - gh - hf - fg = (f + \rho g + \rho^2 h)(f + \rho^2 g + \rho h),$$

where ρ is one of the imaginary cube roots of unity, we have

$$\begin{aligned} -S^3 &= (f + \rho g + \rho^2 h)^3 (f + \rho^2 g + \rho h)^3 \\ &= \{f^3 + g^3 + h^3 + 6fgh + 3\rho(g^2h + h^2f + f^2g) + 3\rho^2(gh^2 + hf^2 + fg^2)\} \\ &\times \{f^3 + g^3 + h^3 + 6fgh + 3\rho^2(g^2h + h^2f + f^2g) + 3\rho(gh^2 + hf^2 + fg^2)\} \end{aligned}$$

whence

$$\begin{aligned} -4S^3 &= \{2(f^3 + g^3 + h^3 + 6fgh) - 3(g^2h + gh^2 + \ldots) + 3\sqrt{-3}(g^2h - gh^2 + \ldots)\} \\ &\times \{2(f^3 + g^3 + h^3 + 6fgh) - 3(g^2h + gh^2 + \ldots) - 3\sqrt{-3}(g^2h - gh^2 + \ldots)\} \\ &= \{2(f^3 + g^3 + h^3) + 12fgh - 3(g^2h + gh^2 + \ldots)\}^2 + 27(g^2h - gh^2 + \ldots)^2. \end{aligned}$$

Also,

$$T^2 = 16\{2(f^3+g^3+h^3)+12fgh-3(g^2h+gh^2+\ldots)\}^2.$$

Hence

$$T^2+64S^3 = -16.27(g^2h-gh^2+h^2f-hf^2+f^2g-fg^2)^2 = -16.27\begin{vmatrix} 1 & 1 & 1 \\ f & g & h \\ f^2 & g^2 & h^2 \end{vmatrix}^2$$

$$= -16.27(g-h)^2(h-f)^2(f-g)^2 = -16.27(\beta^2-\gamma^2)^2(\gamma^2-\alpha^2)^2(\alpha^2-\beta^2)^2.$$

Hence, in order that the cubic shall have a double point, it is necessary and sufficient that $\beta^2=\gamma^2$ or $\gamma^2=\alpha^2$ or $\alpha^2=\beta^2$, *i. e.* that the point α, β, γ lie on one of the lines $y \pm z$, $z \pm x$, $x \pm y$; that is, on one of the degenerate conics of the system. But when $\beta=\gamma$, the equation of the cubic becomes

$$(y-z)\{\beta(x^2+yz)-\alpha x(y+z)\}=0,$$

and similarly for the other cases; hence the cubic cannot have a double point without breaking up into a straight line and a conic.

The discriminant of $\beta(x^2+yz)-\alpha x(y+z)$ is $\frac{1}{4}\beta(\alpha^2-\beta^2)$, which vanishes when $\beta=0$ or $\beta=\pm\alpha$; in the latter case P is one of the four base-points, in the former case it is one of the three centres; these, then, are the only positions of P which give rise to a cubic consisting of three straight lines.

It was obvious geometrically that if P were taken on one of the degenerate conics, that branch of the degenerate conic on which it lay would be part of the locus, which would thus consist of a straight line and a conic; and that if P were one of the four base-points or one of the three centres the locus would consist of three straight lines; the algebraic investigation has served to show that these are the only cases of singularity in the cubic.